Fuel Optimization

First we import the necessary packages: `cvxpy` for solving optimization problems, `numpy` for linear algebra and `matplotlib` for using the `stairs()` function to graph the signal.

[1]:
```
import cvxpy as cp
import numpy as np
import matplotlib.pyplot as plt
```

Next we enter the data required to form the optimization problem, namely A, b, x_{des}, N.

[2]:
```
A = np.matrix([[-1, 0.4, 0.8],[1, 0, 0],[0, 1, 0]])
b = [1, 0, 0.3]
x_des = [7, 2, -6]
N = 30
```

Solving the original problem

The original optimization problem is

$$\min_u \sum_{i=0}^{N-1} f(u(i))$$
$$\text{s.t. } Mu = x_{\text{des}}$$

where

$$M = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \cdots & Ab & b \end{bmatrix}.$$

In the next chunk of code, we compute M by iteratively acting A on b and pushing them as columns of M. We also initialize a list `c = [1,1,...,1]` which essentially does the job of an $N \times 1$ column vector comprising all 1's.

[3]:
```
M = []
temp = np.matrix([[1,0,0],[0,1,0],[0,0,1]])
for i in range(N):
    M.insert(0, (temp @ b).tolist()[0])
    temp = A * temp
M = np.matrix(M).T
```

```
c = [1] * N
```

Now we want to focus on the function $f(a) = \begin{cases} |a| & \text{if } |a| \leq 1 \\ 2|a| - 1 & \text{otherwise} \end{cases}$. It is easy to see that this expression is exactly $\max(|a|, 2|a| - 1)$. So each summand in the abovementioned objective is precisely $\max(|u(i)|, 2|u(i)| - 1)$. This is the expression `f` (so `f[i] = max(abs(u[i]), abs(2u[i])-1)`) in the following block of code. The objective `obj` is simply the sum of `f[i]`'s and the only constraint `cons` is $Mu = x_{\text{des}}$.

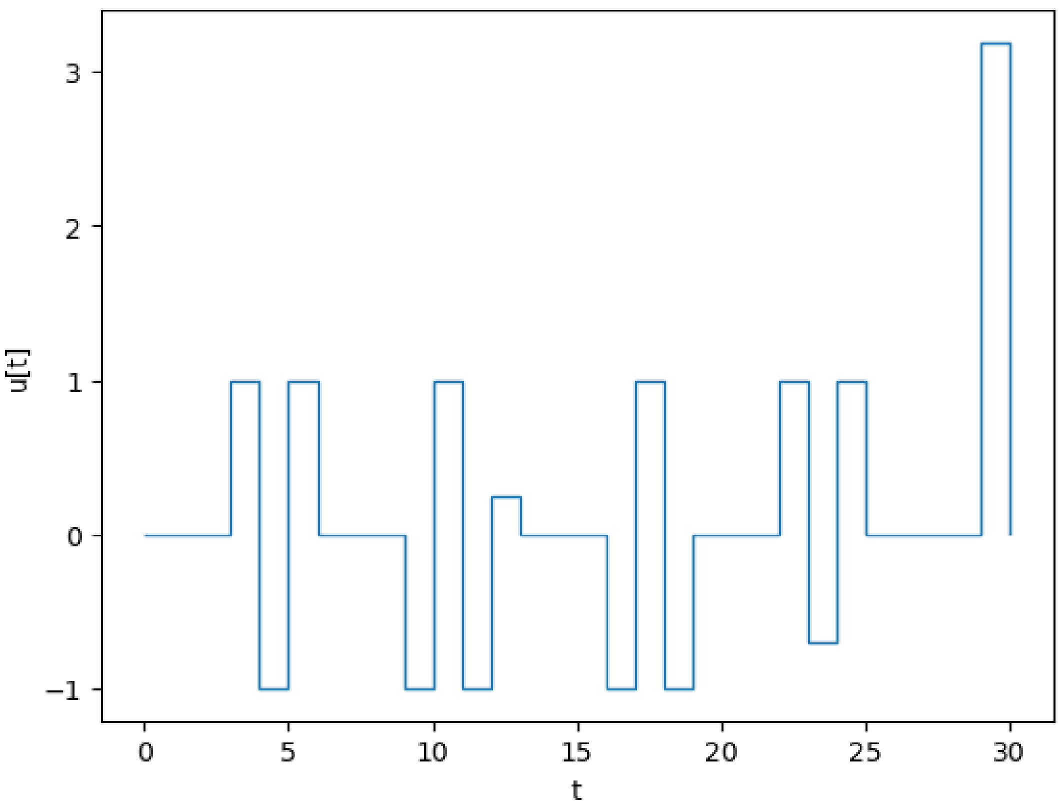
[4]:
```
u = cp.Variable(N, 'u')
f = cp.maximum(cp.abs(u), cp.abs(u+u)-c)
obj = cp.sum(f)
cons = [M @ u == x_des]
problem = cp.Problem(cp.Minimize(obj), cons)
problem.solve(verbose = True, solver = cp.ECOS)
```

```
===============================================================================
                                     CVXPY
                                     v1.4.2
===============================================================================
(CVXPY) Feb 24 04:11:56 PM: Your problem has 30 variables, 1 constraints, and 0
parameters.
(CVXPY) Feb 24 04:11:56 PM: It is compliant with the following grammars: DCP,
DQCP
(CVXPY) Feb 24 04:11:56 PM: (If you need to solve this problem multiple times,
but with different data, consider using parameters.)
(CVXPY) Feb 24 04:11:56 PM: CVXPY will first compile your problem; then, it will
invoke a numerical solver to obtain a solution.
(CVXPY) Feb 24 04:11:56 PM: Your problem is compiled with the CPP
canonicalization backend.
-------------------------------------------------------------------------------
                                  Compilation
-------------------------------------------------------------------------------
(CVXPY) Feb 24 04:11:56 PM: Compiling problem (target solver=ECOS).
(CVXPY) Feb 24 04:11:56 PM: Reduction chain: Dcp2Cone -> CvxAttr2Constr ->
ConeMatrixStuffing -> ECOS
(CVXPY) Feb 24 04:11:56 PM: Applying reduction Dcp2Cone
(CVXPY) Feb 24 04:11:56 PM: Applying reduction CvxAttr2Constr
(CVXPY) Feb 24 04:11:56 PM: Applying reduction ConeMatrixStuffing
(CVXPY) Feb 24 04:11:56 PM: Applying reduction ECOS
(CVXPY) Feb 24 04:11:56 PM: Finished problem compilation (took 1.030e-02
seconds).
-------------------------------------------------------------------------------
                                Numerical solver
-------------------------------------------------------------------------------
```

```
(CVXPY) Feb 24 04:11:56 PM: Invoking solver ECOS  to obtain a solution.
-------------------------------------------------------------------------------
                                    Summary
-------------------------------------------------------------------------------
(CVXPY) Feb 24 04:11:56 PM: Problem status: optimal
(CVXPY) Feb 24 04:11:56 PM: Optimal value: 1.732e+01
(CVXPY) Feb 24 04:11:56 PM: Compilation took 1.030e-02 seconds
(CVXPY) Feb 24 04:11:56 PM: Solver (including time spent in interface) took
1.340e-03 seconds
```

[4]: 17.32356785630167

[5]:
```
plt.stairs(u.value, range(N+1))
plt.xlabel('t')
plt.ylabel('u[t]')
plt.show()
```



Solving the proposed LP

The above solution was based on directly solving the optimization problem, without writing it as an LP. We now solve the following LP which is mentioned in my solution:

$$\begin{aligned} \min_{q,v,w} & \sum_{i=0}^{N-1} w(i) \\ \text{s.t.} \;\; & Mq = x_{\text{des}} \\ & -v + q \leq 0 \\ & -v - q \leq 0 \\ & -w + v \leq 0 \\ & 2v - \boldsymbol{c} - w \leq 0. \end{aligned}$$

Note that the variable u has been replaced with q because we used u above.

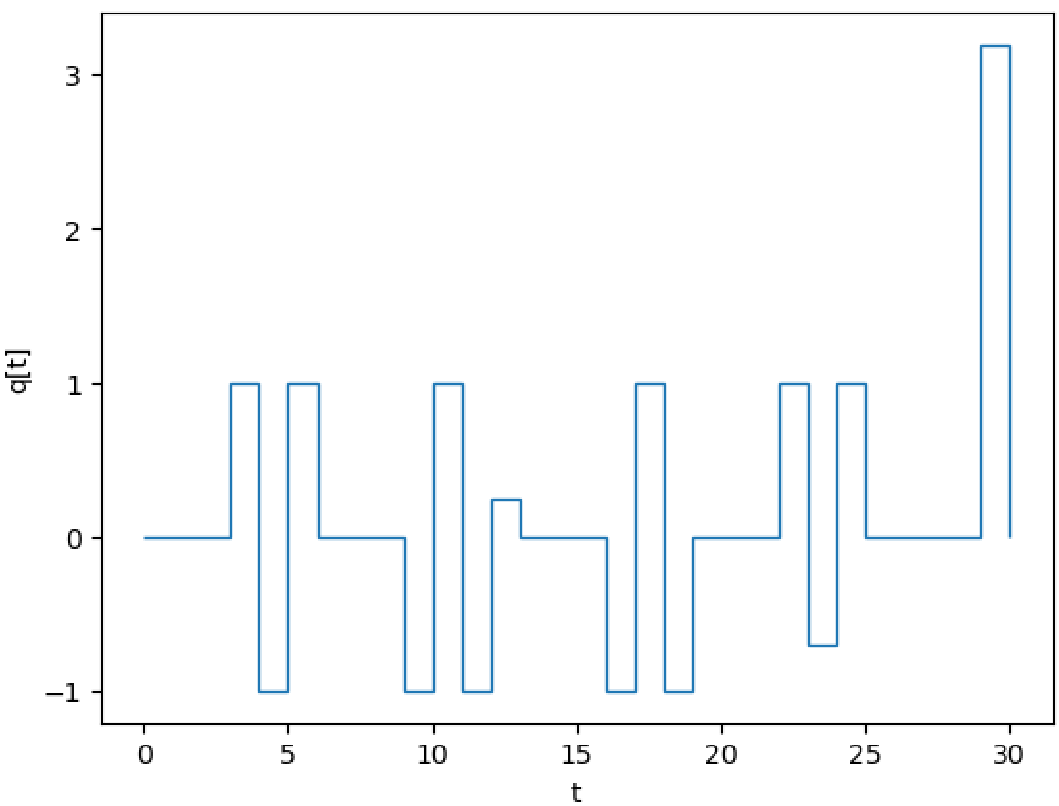
[6]:
```
q = cp.Variable(N, 'q')
v = cp.Variable(N, 'v')
w = cp.Variable(N, 'w')
ob = cp.sum(w)
con = [M @ q == x_des, q - v <= 0, q + v >= 0, v - w <= 0, v + v - c - w <= 0]
pb = cp.Problem(cp.Minimize(ob), con)
pb.solve(verbose = True, solver = cp.ECOS)
```

```
===============================================================================
                                     CVXPY
                                     v1.4.2
===============================================================================
(CVXPY) Feb 24 04:11:56 PM: Your problem has 90 variables, 5 constraints, and 0
parameters.
(CVXPY) Feb 24 04:11:56 PM: It is compliant with the following grammars: DCP,
DQCP
(CVXPY) Feb 24 04:11:56 PM: (If you need to solve this problem multiple times,
but with different data, consider using parameters.)
(CVXPY) Feb 24 04:11:56 PM: CVXPY will first compile your problem; then, it will
invoke a numerical solver to obtain a solution.
(CVXPY) Feb 24 04:11:56 PM: Your problem is compiled with the CPP
canonicalization backend.
-------------------------------------------------------------------------------
                                  Compilation
-------------------------------------------------------------------------------
(CVXPY) Feb 24 04:11:56 PM: Compiling problem (target solver=ECOS).
(CVXPY) Feb 24 04:11:56 PM: Reduction chain: Dcp2Cone -> CvxAttr2Constr ->
ConeMatrixStuffing -> ECOS
(CVXPY) Feb 24 04:11:56 PM: Applying reduction Dcp2Cone
(CVXPY) Feb 24 04:11:56 PM: Applying reduction CvxAttr2Constr
(CVXPY) Feb 24 04:11:56 PM: Applying reduction ConeMatrixStuffing
(CVXPY) Feb 24 04:11:56 PM: Applying reduction ECOS
```

```
(CVXPY) Feb 24 04:11:56 PM: Finished problem compilation (took 1.040e-02
seconds).
-------------------------------------------------------------------------------
                              Numerical solver
-------------------------------------------------------------------------------
(CVXPY) Feb 24 04:11:56 PM: Invoking solver ECOS  to obtain a solution.
-------------------------------------------------------------------------------
                                  Summary
-------------------------------------------------------------------------------
(CVXPY) Feb 24 04:11:56 PM: Problem status: optimal
(CVXPY) Feb 24 04:11:56 PM: Optimal value: 1.732e+01
(CVXPY) Feb 24 04:11:56 PM: Compilation took 1.040e-02 seconds
(CVXPY) Feb 24 04:11:56 PM: Solver (including time spent in interface) took
8.488e-04 seconds
```

[6]: 17.323567854988987

[7]:
```python
plt.stairs(q.value, range(N+1))
plt.xlabel('t')
plt.ylabel('q[t]')
plt.show()
```



The LP formulation indeed gives the same solution as the original formulation. The following line determines the ℓ_1 error between the optimal solutions of the above two problems, which is $< 10^{-8}$, so the solutions are practically the same.

[8]:
```
print(sum([abs(q.value[i] - u.value[i]) for i in range(N)]))
```

```
7.965044064222433e-09
```